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Rigorous Estimates of the Tails of the Probability Distribution Function for the Random Linear Shear Model

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In previous work Majda and McLaughlin, and Majda computed explicit expressions for the $2N$th moments of a passive scalar advected by a linear shear flow in the form of an integral over $\mathbf{R}^N$. In this paper we first compute the asymptotics of these moments for large moment number. We are able to use this information about the large-N behavior of the moments, along with some basic facts about entire functions of finite order, to compute the asymptotics of the tails of the probability distribution function. We find that the probability distribution has Gaussian tails when the energy is concentrated in the largest scales. As the initial energy is moved to smaller and smaller scales we find that the tails of the distribution grow longer, and the distribution moves smoothly from Gaussian through exponential and "stretched exponential." We also show that the derivatives of the scalar are increasingly intermittent, in agreement with experimental observations, and relate the exponents of the scalar derivative to the exponents of the scalar.

KEY WORDS: Passive scalar intermittency; turbulence; Hamburger moment problem.

1. INTRODUCTION

It is a well documented experimental fact that, while the statistics of the velocity field in a turbulent flow are roughly Gaussian, the statistics of

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other quantities like the pressure, derivatives of velocity and a passively adverted scalar are generally far from Gaussian.[7, 8, 15, 19, 23, 50] For example Castaing *et al.*[8] observed in experiments in a Rayleigh–Bénard convection cell that for Rayleigh number $Ra < 10^7$ the distribution of temperature appeared to be roughly Gaussian, while for larger Rayleigh numbers, $Ra > 10^8$, the temperature distribution appeared to be closer to exponential. In related work Ching[15] studied the probability distribution functions (pdfs) for temperature differences at different scales, again in a Rayleigh–Bénard cell, and found that the pdfs over a wide range of scales were well approximated by a "stretched exponential" distribution of the form

$$P(T) = e^{-C\,|T|^{\beta}}$$

At the smallest scales the observed value of the exponent was $\beta \approx 0.5$, while at the largest scales the observed exponent was roughly $\beta \approx 1.7$. Kailasnath, Sreenivasan and Stolovitky[23] measured the pdfs of velocity differences in the atmosphere for a wide range of separation scales. They found similar distributions to the ones found by Ching, with exponents ranging from $\beta \approx 0.5$ for separation distances in the dissipative range to $\beta \approx 2$ on the integral scale. Finally Thoroddsen and Van Atta[50] studied thermally stratified turbulence in a wind tunnel and found the probability distributions of the density to be roughly Gaussian, while the distributions of the density gradients were exponential.

A complete understanding of such intermittency lies at the heart of understanding fluid turbulence, and would certainly require a detailed understanding of the creation of small scale fluid structures involving both patchy regions of strong vorticity and intense gradients.[17, 49] An alternative starting point is to assume the statistics of the flow are known a priori and to determine how these statistics are manifest in a passively evolving quantity. This question of inherited statistics is significantly easier than the derivation of a complete theory for fluid turbulence, though still retains many inherent difficulties such as problems of closure.

Motivated by the Chicago experiments of the late 80's,[8] and earlier work,[2, 28, 36, 48] there has been a tremendous effort towards understand the origin of the intermittent temperature probability distribution function in passive scalar models with prescribed (usually Gaussian) velocity statistics. For a very complete review of the subject of turbulent diffusion, including a full discussion of scalar intermittency see the recent survey article of Majda and Kramer.[31] Most of the work on the scalar statistics has either been directed at understanding the anomalous scaling of temperature structure functions, or at understanding the shape of the tail of the limiting scalar pdf.

There has been a wealth of theoretical efforts addressing this last issue of the tail.[3, 5, 8, 10, 11, 14, 16, 18, 20, 34, 25, 24, 29, 30, 32, 37, 38, 43, 45, 46, 52] A somewhat common theme, particularly in the pumped case, is the prediction that the scalar pdf should develop an exponential tail. For example Kraichnan[25] Shraiman and Siggia[43] and Balkovsky and Falkovich[3] all find exponential tails. Another important question is to understand the pdf of the scalar gradient. Naturally, gradient information may be expected to amplify contributions from small scales, and a general theory relating the scalar tail with the gradient tail, even for passively evolving quantities would be quite valuable. There has been somewhat less theoretical effort aimed at exploring the difference in statistics between the scalar and the scalar gradient. Chertkov, Falkovich and Kolokolov,[12] Chertkov Kolokolov and Vergassola[13] and Balkovsky and Falkovich[3] have explored this question and have found a stretched exponential distribution of the scalar gradient in situations for which the scalar has an exponential tail. Holzer and Siggia,[20, 21] and Chen and Kraichnan[9] have observed similar phenomena numerically.

In this paper we examine the scalar and scalar gradient pdf tail in an exactly solvable model first studied by Majda[29] and McLaughlin and Majda[32] who were able to construct explicit moment formulas for the moments of a passive scalar adverted by a rapidly fluctuating linear shear flow in terms of N-dimensional integrals. In that work, it was established that the degree of length scale separation between the initial scalar field and the fluid flow is inherent to the development of a broader than Gaussian pdf.

Here, we explicitly calculate the tails of the pdf for this model. We begin by analyzing the expression derived by Majda for the large time $2N$th moment of the pdf for the random uniform shear model, which is given by an integral over $\mathbf{R}^N$. From these normalized moments, we will construct the tail of the associated pdf. We point out that in this calculation the convergence of the pdf for finite time to the pdf for infinite time is weak—for fixed moment number the finite time moment converges to the limiting moment. The convergence is almost certainly not uniform in the moments. For a more thorough investigation of the uniformity of this limiting process in the context of general, bounded periodic shear layers, see Bronski and McLaughlin.[5]

The tail is calculated in two steps. First, using direct calculation and gamma function identities we are able to reduce the N-dimensional integral to a *single* integral of Laplace type, from which the asymptotic behavior of the $2N$th moment follows easily. The asymptotic behavior of the moments is important for determining the tails of the probability distribution function, as we establish below. Second, we consider the problem of reconstructing the probability measure from the moments. Using ideas from complex analysis, mainly some basic facts about entire functions of finite order and type, we are able to provide rigorous estimates for the rate of decay of the tails of the measure. We find that the tails decay like

$$\exp(-c_\alpha |T|^{4/(3+\alpha)})$$

so depending on the precise value of the parameter α (defined in Section II, below, which sets the degree of scale separation between the scalar and flow field) the model admits tails which are Gaussian, exponential, or stretched exponential. We also show that in this model higher order derivatives of the scalar in the shear direction are always more intermittent, with a very simple relationship between the exponents of the scalar and its derivative. The distributions of derivatives in the cross-shear direction, however, display the same tails as the scalar itself.

We remark that, while the stream-line topology for shear profiles is admittedly much simpler than that in fully developed turbulence, the fact that the exact limiting tail for the decaying scalar field may be explicitly and rigorously constructed suggests such models to be exceptionally attractive for testing the validity of different perturbation schemes. It is also extremely interesting because it demonstrates that, at least for unbounded flows, a positive Lyapunov exponent (as would typically occur for a general Batchelor flow) is *not necessary* for intermittency. For an interesting discussion of the role of Lyapunov exponents in producing intermittency see the work of Chertkov, Falkovich, Kolokolov and Lebedev.[14]

A. The Random Shear Model

Here, we briefly review the framework of the random shear model.[29, 30, 32, 5] We follow Majda, and consider the free evolution of a passive scalar field in the presence of a rapidly fluctuating shear profile:

$$\frac{\partial T}{\partial t} + \gamma(t)\, v(x)\, \frac{\partial T}{\partial y} = \bar{\kappa}\, \Delta T \tag{1}$$

The random function, $\gamma(t)$, represents multiplicative, mean zero Gaussian white noise, delta correlated in time:

$$\langle \gamma(t)\, \gamma(s) \rangle = \delta(|t-s|)$$

where the brackets, $\langle \cdot \rangle$, denote the ensemble average over the statistics of $\gamma(t)$. The original model considered by Majda, involved the case of a uniform shear layer, $v(x) = x$, which leads to the moments considered below.[29] It a quite general fact, not special to shear profiles, that a closed evolution equation for the arbitrary N-point correlator is available for the special case of rapidly fluctuating Gaussian noise, see work of Majda[30] for a path integral representation of this fact for the special case of random shear layers. For the scalar evolving in (1), the N point correlator, defined as:

$$\psi_N(\mathbf{x}, \mathbf{y}, t) = \left\langle \prod_{j=1}^{N} T(x_j, y_j, t) \right\rangle$$
$$\mathbf{x} = (x_1, x_2, x_3, ..., x_N) \tag{2}$$
$$\mathbf{y} = (y_1, y_2, y_3, ..., y_N)$$

is a function: ψ_N: $R^{2N} \times [0, \infty) \to R^1$ satisfying

$$\frac{\partial \psi_N}{\partial t} = \bar{\kappa}\, \Delta_{2N} \psi_N + \frac{1}{2} \sum_{i,\, j=1}^{N} v(x_i)\, v(x_j) \frac{\partial^2 \psi_N}{\partial y_i\, \partial y_j} \tag{3}$$

where Δ_{2N} denotes the $2N$ dimensional Laplacian.

We next describe the initial scalar field. Following Majda,[29] we assume that the scalar is initially a mean zero, Gaussian random function depending only upon the variable, y:

$$T|_{t=0} = \int_{R^1} dW(k)\, e^{2\pi i k y}\, |k|^{\alpha/2}\, \hat{\phi}_0(k) \qquad \alpha > -1 \tag{4}$$

Here, $\hat{\phi}_0(k)$ denotes a rapidly decaying (large k) cut-off function satisfying $\hat{\phi}_0(k) = \hat{\phi}(-k)$, $\hat{\phi}_0(0) \neq 0$ and dW denotes complex Gaussian white noise with

$$\langle dW \rangle_W = 0$$
$$\langle dW(k)\, dW(\eta) \rangle_W = \delta(k + \eta)\, dk\, d\eta$$

The spectral parameter, α appearing in (4) is introduced to adjust the excited length scales of the initial scalar field, with increasing α corresponding to initial data varying on smaller scales. We remark that the more general case involving initial data depending upon both x and y, and data possessing both mean and fluctuating components, was analyzed by McLaughlin and Majda.[32]

For this case involving shear flows, the evolution of this N point correlator may be immediately converted to parabolic quantum mechanics through partial Fourier transformation in the $\mathbf{y}$ variable. For the particular initial data presented in (4), this yields the following solution formula:

$$\psi_N = \int_{R^N} e^{2\pi i \mathbf{k} \cdot \mathbf{y}} \hat{\psi}_N(\mathbf{x}, \mathbf{k}, t) \prod_{j=1}^{N} \hat{\phi}_0(k_j)\, |k_j|^{\alpha/2}\, dW(k_j)$$

where the N-body wavefunction, $\hat{\psi}_N(\mathbf{x}, \mathbf{k}, t)$ satisfies the following parabolic Schrödinger equation:

$$\begin{aligned} \frac{\partial \hat{\psi}_N}{\partial t} &= (\bar{\kappa}\, \Delta_{\mathbf{x}} - V_{\text{int}}(\mathbf{k}, \mathbf{x}))\, \hat{\psi}_N \\ \hat{\psi}_N|_{t=0} &= 1 \end{aligned} \tag{5}$$

The interaction potential, $V_{\text{int}}(\mathbf{k}, \mathbf{x})$, is

$$V_{\text{int}} = 4\pi^2\, |\mathbf{k}|^2 + 2\pi^2 \left(\sum_{j=1}^{N} k_j v(x_j) \right)^2$$

For the special case of a uniform, linear shear profile, with $v(x) = x$, the quantum mechanics problem in (5) is exactly solvable in any spatial dimension. Taking the ensemble average over the initial Gaussian random measure using a standard cluster expansion, the general solution formula for $\langle \psi_N(\mathbf{x}, \mathbf{y}, t) \rangle_W$ is obtained[29, 32] in terms of N dimensional integrals. The normalized, long time flatness factors, $\mu_{2N}^{\alpha} = \lim_{t \to \infty} (\langle T^{2N} \rangle / \langle T^2 \rangle^N)$, are calculated by evaluating the correlator along the diagonal,

$$\mathbf{x} = (x, x, x, ..., x)$$

$$\mathbf{y} = (y, y, y, ..., y)$$

and utilizing the explicit long time asymptotics available through Mehler's formula. This leads to the following set of normalized moments for the decaying scalar field, T:

$$\begin{aligned} \mu_{2N}^{\alpha} &= \frac{(2N)!}{2^N N!\, \sigma^N} \int_{R^N} d\mathbf{k}\, \frac{\prod_{j=1}^{N} |k_j|^{\alpha}}{\sqrt{\cosh(|\mathbf{k}|)}} \\ \sigma &= \int_{R^1} dk\, \frac{|k|^{\alpha}}{\sqrt{\cosh |k|}} \end{aligned} \tag{6}$$

Observe that these normalized moments depend upon the parameter α. By varying this parameter Majda and McLaughlin established that the degree of scale separation between the initial scalar and flow field is important in the development of a broader than Gaussian pdf.[29, 32] They demonstrated through numerical quadrature of these integrals for low order moments that as the initial scalar field develops an infrared divergence (with $\alpha \to -1$, corresponding to the loss of scale separation between the initial scalar field, and the infinitely correlated linear shear profile) the limiting single point scalar distribution has Gaussian moments.[29] Conversely they showed that as the length scale of the initial scalar field is reduced, corresponding to increasing values of α, the limiting distribution shows growing moments indicative of a broad tailed distribution.[32] On the basis of these low order moment comparisons, these studies suggest that within these models, the limiting pdf should be dependent upon the scale separation between the scalar and flow field. A fundamental issue concerns whether and how this scale dependence is manifest in the pdf *tail*. Below, we address precisely this issue, and rigorously establish that the intuition put forth by Majda and McLaughlin is correct through the explicit calculation of the limiting pdf tail.

II. ASYMPTOTICS OF THE PROBABILITY DISTRIBUTION

A. Notation

Recall from the previous section that the work of Majda derived exact expressions for the moments of a one parameter family of models indexed by the exponent α. In the remainder of the paper $d\mu^\alpha(T)$ will denote the probability measure for the passive scalar T in the Majda model with exponent α. The ith moment of the probability measure $d\mu^\alpha(T)$ will be denoted by μ_i^α. In this particular model the distribution is symmetric and thus all odd moments vanish.

B. Large-N Asymptotics of the Moments

In this model the exact expression for the $2N$th moment is given by

$$\mu_{2N}^\alpha = \frac{(2N)!}{\sigma^N 2^N N!} \int \frac{\prod_{j=1}^N |k_j|^\alpha}{\sqrt{\cosh(|\vec{k}|)}}\, dk_1\, dk_2\, dk_3 \cdots dk_N$$

$$\sigma = \int \frac{|k|^\alpha\, dk}{\sqrt{\cosh(k)}}$$

As noted by Majda $\cosh(|\vec{k}|) \leqslant \prod \cosh(k_i)$ which implies the normalized flatness factors are strictly larger than those of a Gaussian, implying broad tails. The simplest way to analyze this integral, and in particular to understand the behavior for large N, is to introduce spherical coordinates. Spherical coordinates in N dimensions can easily be constructed iteratively in terms of spherical coordinates in $N-1$ dimensions as follows. The coordinates in N dimensional spherical coordinates are $\{r, \theta_1, \theta_2, \theta_3, \ldots, \theta_{N-1}\}$. If $\{x_1^{N-1}, x_2^{N-1}, \ldots, x_{N-1}^{N-1}\}$ are coordinates on $\mathbf{R}^{N-1}$ then coordinates on $\mathbf{R}^N$ are given by

$$\begin{aligned} x_j^N &= x_j^{N-1} \sin(\theta_{N-1}) \qquad j \in 1 \cdots N-1 \\ x_N^N &= r \cos \theta_{N-1} \end{aligned}$$

Using this construction it is simple to calculate that the volume element in N dimensional spherical coordinates is given by

$$\begin{aligned} &dx_1\, dx_2 \cdots dx_N \\ &\quad = r^{N-1}\, dr \prod_{j=1}^{N-1} \sin^{j-1}(\theta_j)\, d\theta_j \qquad \theta_1 \in [0, 2\pi] \qquad \theta_{i>1} \in [0, \pi] \end{aligned}$$

Since the volume element is a product measure the N dimensional integral factors as a product of N one dimensional integrals and we are left with the expression

$$\mu_{2N}^{\alpha} = \frac{(2N)!}{\sigma^N 2^N N!}\, I_0(N) \prod_{j=1}^{N-1} I_j$$

where the I_j are given by

$$\begin{aligned} I_0(N) &= \int_0^{\infty} r^{N(\alpha+1)-1} \frac{dr}{\sqrt{\cosh(r)}} \\ I_1 &= \int_0^{2\pi} |\sin(\theta)|^{\alpha}\, |\cos(\theta)|^{\alpha}\, d\theta \\ I_j &= \int_0^{\pi} |\sin(\theta)|^{j(\alpha+1)-1}\, |\cos(\theta)|^{\alpha}\, d\theta \qquad j > 1 \end{aligned} \tag{7}$$

The angular integrals can be done explicitly in terms of gamma functions, using the beta function identity

$$2 \int_0^{\pi/2} |\sin(\theta)|^{2z-1}\, |\cos(\theta)|^{2w-1}\, d\theta = \beta(z, w) = \frac{\Gamma(z)\, \Gamma(w)}{\Gamma(z+w)}$$

which leads to the expression

$$\mu_{2n}^{\alpha} = 2\,\frac{(2N)!}{\sigma^N 2^N N!}\,I_0(N)\prod_{j=1}^{N-1}\frac{\Gamma\left(\left(\frac{\alpha+1}{2}\right)\Gamma\left(j\frac{\alpha+1}{2}\right)\right)}{\Gamma\left((j+1)\frac{\alpha+1}{2}\right)}$$

$$= 2\,\frac{(2N)!\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^{N-1}}{\sigma^N 2^N N!}\,I_0(N)\prod_{j=1}^{N-1}\frac{\Gamma\left(j\frac{\alpha+1}{2}\right)}{\Gamma\left((j+1)\frac{\alpha+1}{2}\right)} \tag{8}$$

Observe that the product telescopes—the numerator of one term is the denominator of the next—giving the final expression

$$\mu_{2n}^{\alpha} = 2\,\frac{(2N)!}{\sigma^N 2^N N!}\,\frac{\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^N}{\Gamma\left(N\frac{\alpha+1}{2}\right)}\int r^{N(\alpha+1)-1}\,\frac{dr}{\sqrt{\cosh(r)}}$$

$$= 2\,\frac{(2N)!}{\sigma^N 2^N N!}\,\frac{\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^N}{\Gamma\left(N\frac{\alpha+1}{2}\right)}\,I_0(N) \tag{9}$$

The integral over the radial variable $I_0(N)$ cannot be done explicitly, but the large N asymptotics are given by

$$I_0(N) \approx 2^{N(\alpha+1)+1/2}\,\Gamma(N(\alpha+1))$$

so that the large N behavior of the moments is given by

$$\mu_{2N}^{\alpha} \approx 2^{N\alpha+3/2}\,\frac{(2N)!}{\sigma^N N!}\,\frac{\Gamma(N(\alpha+1))\left(\Gamma\left(\frac{\alpha+1}{2}\right)\right)^N}{\Gamma\left(N\left(\frac{\alpha+1}{2}\right)\right)} \tag{10}$$

Note that since

$$\frac{\Gamma(N(\alpha+1))}{\Gamma\left(\frac{N(\alpha+1)}{2}\right)} \to \infty \qquad \text{as} \quad N \to \infty$$

the moments are strictly larger than the moments of the Gaussian. We will use this to provide rigorous quantitative estimates for the tails of the distribution.

C. The Hamburger Moment Problem

Having computed simple expressions for the moments of the pdf, as well as asymptotic expressions for large moment number, it is natural to ask the question of whether one can do the inverse problem and deduce the pdf itself. The problem of determining a measure from its moments is a classical one, known as the Hamburger moment problem.[39, 42, 51] This problem has a rich theory, and we mention only a vary few of the most basic results here. For an overview of the subject, see the book by Shohat and Tamarkin[42] or the recent electronic preprint by Simon.[44]

The two most important questions are, of course, existence and uniqueness. There is a necessary and sufficient condition for a set of numbers $\{\mu_i\}$ to be the moments of some probability measure, namely that the expectation of any positive polynomial be positive. This translates into the following linear algebraic conditions on the diagonal determinants of the Hankel matrix, the matrix with i, jth entry μ_{i+j}:

$$|\mu_0| > 0, \qquad \begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix} > 0, \qquad \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix} > 0 \cdots$$

These conditions appear to be quite difficult to check in practice. However since the moments considered here are, by construction, the moments of a pdf this condition must hold.

A more subtle question is the issue of uniqueness of the measure, usually called determinacy in the literature of the moment problem. One classical sufficient condition for the determinacy of the moment problem is the following condition, due to Carleman:[6, 42] If the moments μ_n are such that the following sum *diverges*

$$\sum_{j=1}^{\infty} (\mu_{2j})^{-1/2j} = \infty$$

then the Hamburger moment problem is determinate. Given the asymptotic expression for the moments given in Eq. (10) it is easy to check that

$$(\mu_{2j}^{\alpha})^{-1/2j} \approx cj^{-(\alpha+3)/4}$$

and thus there is a unique measure with these moments for $-1 \leqslant \alpha \leqslant 1$. We will see later that this corresponds to probability distribution functions with tails that range from Gaussian through exponential.

In the case $\alpha > 1$ which, as we will see later, corresponds to stretched exponential tails, the problem probably does not have a unique solution. Indeed there are classical examples of collections of moments with the same asymptotic behavior as the stretched exponential distribution for which the moment problem has a whole family of solutions.

Given this we come to the question of actually calculating the measure given the moments. There is a rather involved theory for this in the determinate case involving, among other things, orthogonal polynomials and continued fractions,[27, 42] but in general this problem is extremely difficult. However we show in the next section that it is relatively straightforward to reconstruct the *tails of the measure* from the moments.

D. Asymptotics of the Tails of the Distribution

Recall that μ_{2N}^{α} is the $2N$th moment of some probability measure $d\mu^{\alpha}(T)$,

$$\mu_{2N}^{\alpha} = \int T^{2N}\, d\mu^{\alpha}(T) \tag{11}$$

We are interested in calculating the asymptotic rate of decay of the tails of the probability measure $d\mu^{\alpha}(T)$. The information about the behavior of the tails of the distribution is contained in the asymptotic behavior of the large moments. We study the tails of the measure $d\mu^{\alpha}(T)$ by introducing the function

$$f^{\alpha}(z) = \sum_{j=0}^{\infty} \frac{\mu_{2j}^{\alpha} z^{2j}}{\Gamma\left(\frac{j(3+\alpha)}{2}\right) C^{2j}} \tag{12}$$

where C is some as yet unspecified constant. The factor of $\Gamma(j(3+\alpha)/2)$ is chosen so that the series for f^{α} has a finite but non-zero radius of convergence. This will give us the sharpest control over the tails of $d\mu^{\alpha}(T)$. It is convenient to demand that the radius of convergence of the series be one. Using the root test it is straightforward to check that the radius of convergence of the sum is given by

$$r^* = C 2^{-(\alpha+2)} \frac{(\alpha+3)^{(\alpha+3)/4}}{(\alpha+1)^{(\alpha+1)/4}} \sqrt{\frac{\sigma}{\Gamma((\alpha+1)/2)}}$$

so we choose

$$C = 2^{\alpha+2} \frac{(\alpha+1)^{(\alpha+1)/4}}{(\alpha+3)^{(\alpha+3)/4}} \sqrt{\frac{\Gamma((\alpha+1)/2)}{\sigma}}$$

Since the coefficients μ^α_{2N} are the moments of a probability measure $d\mu^\alpha(T)$ we have the alternative expression

$$f^\alpha(z) = \sum_{j=0}^{\infty} \frac{z^{2j}}{C^{2j}\Gamma\left(\frac{j(3+\alpha)}{2}\right)} \int T^{2j}\, d\mu^\alpha(T)$$

When z is inside the radius of convergence of the sum (i.e., $|z|<1$) we can switch the integration and the summation and get the following expression for f^α

$$f^\alpha(z) = \int \sum_{j=0}^{\infty} \frac{T^{2j}z^{2j}}{C^{2j}\Gamma\left(\frac{j(3+\alpha)}{2}\right)}\, d\mu^\alpha(T) \tag{13}$$

$$= \int F^\alpha(zT)\, d\mu^\alpha(T) \tag{14}$$

We note a few simple facts. First notice that the function $f^\alpha(z)$ is a kind of generalized Laplace transform of the measure $d\mu^\alpha(T)$. The quantity inside the integral, $F^\alpha(zT) = \sum (T^{2j}z^{2j})/(C^{2j}\Gamma(j(3+\alpha)/2)$ converges absolutely for all z and thus $F^\alpha(zT)$ is an entire function of the complex variable z. Further we know that the integral must converge for $|z|<1$ and diverge for some $|z|>1$, since the original series converged in a circle of unit radius. We note that the entire function $F^\alpha(z)$ satisfies

$$|F^\alpha(z)| = \left| \sum_{j=0}^{\infty} \frac{z^{2j}}{C^{2j}\Gamma\left(j\frac{3+\alpha}{2}\right)} \right| \tag{15}$$

$$\leqslant \sum_{j=0}^{\infty} \frac{|z|^{2j}}{\left|C^{2j}\Gamma\left(j\frac{3+\alpha}{2}\right)\right|} \tag{16}$$

$$\leqslant F^\alpha(|z|), \tag{17}$$

so the function $F^{\alpha}(z)$ grows fastest along the real axis. Thus we know that the integral in Eq. (14) converges for $-1<z<1$ and diverges for $z>1$, $z<-1$. Thus the problem of understanding the rate of decay of the tails of the probability measure $d\mu^{\alpha}(T)$ has been reduced to that of determining the rate of growth of the function $F(zt)$. There is a well-developed theory for studying the rate of growth of entire functions, the theory of entire functions of finite order. We recall only the basic facts here—the interested reader is referred to the texts of Ahlfors[1] and Rubel with Colliander.[40]

The radial maximal function $M_F(r)$ of an entire function $F(z)$ is defined to be the maximum of the absolute value of F over a ball of radius r centered on the origin:

$$M_F(r)=\max_{|z|=r}|F(z)|$$

The order ρ of a function F is defined to be

$$\rho=\limsup_{r\to\infty}\frac{\log_+\log_+ M_F(r)}{\log_+(r)}$$

where $\log_+(x)=\max(0,\log(x))$, if this limit exists. It is easy to see from this definition that F is of order ρ means that F grows asymptotically like $\exp(A(z)\,|z|^{\rho})$ along the direction of maximum growth in the complex plane, where $A(z)$ grows more slowly than any power of z. A related notion is the type of a function of finite order. If F is of order ρ then the type τ is defined to be

$$\tau=\limsup_{r\to\infty}\frac{\log_+ M_F(r)}{r^{\rho}}$$

when this limit exists. Again speaking very roughly the type τ gives the next order asymptotics: if F is of order ρ and type τ then F grows like $B(z)\exp(\tau\,|z|^{\rho})$, where $B(z)$ is subdominant to the exponential term. Note that by Eq. (17) the function F^{α} grows fastest along the real axis, and thus the maximal rate of growth in the complex plane is exactly the rate of growth along the real axis.

There exist alternate characterizations of the order and type of a function in terms of the Taylor coefficients A_n which are very useful for our purposes. These are given as follows:

$$\rho = \limsup_{r\to\infty} \frac{\log_+ \log_+ M_F(r)}{\log_+ (r)} = \limsup_{n\to\infty} \frac{n\log(n)}{-\log(|A_n|)} \tag{18}$$

$$\tau = \limsup_{r\to\infty} \frac{\log_+ M_F(r)}{r^\rho} = \frac{1}{\rho e} \limsup_{n\to\infty} n\,|A_n|^{\rho/n} \tag{19}$$

For the proofs we refer to the text of Rubel with Colliander.[40] Using the expressions given in Eqs. (18) and (19) we find that the order ρ and type τ of $F^\alpha(z)$ are given by

$$\rho^\alpha = \limsup_{n\to\infty} \frac{2n\log(2n)}{\log\left(C^{2n}\Gamma\left(\frac{(3+\alpha)n}{2}\right)\right)} = \frac{4}{3+\alpha}$$

$$\tau^\alpha = \frac{1}{\rho e} \limsup_{n\to\infty} n\left|\Gamma\left(\frac{(3+\alpha)\,n}{2}\right)\right|^{-\rho/n} = \frac{1}{C^\rho}$$

Thus we know that $F^\alpha(zT)$ grows like $A(zT)\exp(C^{-\rho}\,|z|^{4/(3+\alpha)}\,|T|^{4/(3+\alpha)})$ along the real axis, where $A(zT)$ grows more slowly than $\exp(D\,|T|^{4/(3+\alpha)})$ for any D. Further we know that the integral

$$\int F^\alpha(zT)\,d\mu^\alpha(T)$$

converges for $|z|<1$ and diverges for $z>1$ or $z<-1$, so to leading order the rate of decay of the measure $d\mu^\alpha(T)$ is given by $\exp(-|C|^{-4/(3+\alpha)}\,|T|^{4/(3+\alpha)})$. It is easy to check that as $\alpha\to-1$ this estimate becomes $\exp(-T^2/4)$, recovering the normalized Gaussian.

This result is probably best restated in terms of the cumulative distribution function, rather than the probability measure. If $P(T,T') = \int_T^{T'} d\mu(T)$, with $T'>T$, then it is easy to show that the above implies that

$$\begin{aligned} \lim_{T\to\infty} \exp(c\,|T|^{4/(3+\alpha)})\,P(T,T') &= 0 \qquad & c<|C|^{-4/(3+\alpha)} \\ &= \infty \qquad & c>|C|^{-4/(3+\alpha)} \end{aligned}$$

regardless of whether the measure is unique.

III. INTERPRETATION AND CONCLUDING REMARKS

Physically the Majda model can be thought of as a model for the behavior of a passive scalar at small scales, when the scale of the flow field is *much larger than the scale of the variations of the scalar*. Recall that the random scalar is given by

$$T(y) = \int |k|^{\alpha/2}\, \hat{\phi}_0(k)\, e^{2\pi iky}\, dW(k) \tag{20}$$

$$\langle T(y)\, T(y') \rangle = \int |k|^{\alpha}\, |\hat{\phi}_0(k)|^2\, e^{2\pi ik(y-y')}\, dk \tag{21}$$

In the limit as α approaches -1 there is an infrared divergence, so that the energy of the scalar is concentrated at larger and larger scales. In this case $4/(3+\alpha) \to 2$, so the normalized distribution function becomes Gaussian, as was originally observed by Majda.

One important fact about this model which we would like to emphasize is that it predicts that higher derivatives of the adverted scalar should be *increasingly intermittent*, a fact which was not strongly emphasized in previous work. Observe that due to the special nature of shear flows the scalar derivative $\partial T/\partial y$ satisfies the same equation as the scalar T with *no additional terms*!. We further note that the initial condition for the derivative of the scalar is given by

$$\frac{\partial T}{\partial y} = \int 2\pi i\, |k|^{\alpha/2}\, k\hat{\phi}_0(k)\, e^{2\pi iky}\, dW(k) \tag{22}$$

$$\left\langle \frac{\partial T}{\partial y} \frac{\partial T}{\partial y'} \right\rangle = 4\pi^2 \int |k|^{\alpha+2}\, |\hat{\phi}_0(k)|^2\, e^{2\pi ik(y-y')}\, dk \tag{23}$$

so the derivative of the scalar has a representation of the same form as the representation of the scalar itself, but with the exponent α increased by two, and a slightly modified $\phi_0(k)$. Recall that the exponent α determines the amount of energy at the largest scales and thus the degree of intermittency, with the tails decaying as $\exp(-T^{4/(3+\alpha)})$. Our calculation shows that increasing α increases the width of the tails of the probability distribution function, *implying that derivatives are more intermittent*! These predictions for the behavior of the tails of the scalar as compared with the scalar gradient are in extremely good agreement with experimental and numerical results. For instance our calculation shows that if the scalar has exponent $\alpha = -1$, so that the probability distribution function of the scalar has Gaussian tails, then the derivative of the scalar has exponent $\alpha = 1$, implying that the distribution of the derivative has *exponential tails*. This agrees quite well with the experiments of Van Atta and Thoroddsen,[50] as just one example, who observe that in turbulent thermally stratified flow that the pdf for the density has Gaussian tails, while the pdf of the density gradient has exponential tails. Similarly if the scalar has exponent $\alpha = 1$, so that the distribution of the scalar itself is exponential, then derivative of the scalar should have exponent $\frac{2}{3}$. This agrees with the calculations of Chertkov, Falkovich and Kolokolov,[12] and Balkovsky and Falkovich[3] also predict exponential tails for the scalar and stretched exponential tails with exponent $\frac{2}{3}$ for the scalar gradient in the Batchelor regime. This also shows reasonably good agreement with the numerical experiments of Holzer and Siggia.[20, 21] In their experiments Holzer and Siggia find that a scalar with exponential tails has a gradient with stretched exponential tails. For large Peclet number the exponent of these stretched exponential tails is in the range of 0.661–0.563.

Of course one can eliminate α entirely, and one finds the following relationship between the distribution of the scalar and the scalar gradient within this model. If T is distributed according to a stretched exponential pdf with exponent ρ, and the gradient T_y according to a stretched exponential pdf with exponent ρ', then ρ, ρ' are related by

$$\frac{1}{2} + \frac{1}{\rho} = \frac{1}{\rho'}$$

It would be extremely interesting to check if this relationship, or some generalization of it, holds in greater generality than shear flows. The above numerical and experimental evidence suggest that this might not be an unreasonable hope.

The distribution of the x, or cross-shear, derivatives can also be calculated using the same explicit representations derived by Majda. Calculations by the authors for deterministic initial data have shown that derivatives in the cross-shear direction have a distribution with the *same* asymptotic behavior as the scalar itself. This is in contrast to the work of Son,[46] and Balkovsky and Fouxon,[4] which predict distributions with very broad tails (all of the higher moments diverge as $t \to \infty$) and predicts the same distribution for the scalar and its derivatives.

We would also like to comment on the relationship between intermittency and the Lyapunov exponents of the underlying flow field. A number of papers have addressed the problem of intermittency in the large Peclet number limit by attempting to relate broader than Gaussian tails to the Lyapnuov exponents of the flow field.[14] It is worth noting that a shear flow does not possess a positive Lyapunov exponent, but as we have shown here a shear flow can generated exponential and stretched exponential tails in the passive scalar. This shows that chaotic behavior in the underlying flow, while probably an important effect in realistic flows, is not necessary for the generation of broad tails and intermittency.

Finally we would like to comment on the rate of approach to the limiting measure in time. The results presented here analyze the infinite time limit of the measure. As mentioned earlier the convergence to this limiting measure is expected to be highly nonuniform. A preliminary calculation by the authors for a special choice of the cut-off function $\hat{\phi}_0(k)$ suggests that for large but finite times the pdf looks like the pdf for the infinite time problem in some core region, with Gaussian tails outside this core region. As time increases the size of this core region demonstrating non-Gaussian statistics grows, and the Gaussian tails get pushed out to infinity. We believe this same picture to hold for any choice of the cut-off function $\hat{\phi}_0(k)$, but more work is needed to establish this fact.

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